

CHARACTERIZATIONS OF WEAKLY COMPACT OPERATORS ON $C_o(T)$

T. V. PANCHAPAGESAN

Dedicated to Professor V. K. Balachandran on the occasion of his seventieth birthday

ABSTRACT. Let T be a locally compact Hausdorff space and let $C_o(T) = \{f : T \rightarrow \mathbb{C}, f \text{ is continuous and vanishes at infinity}\}$ be provided with the supremum norm. Let $\mathcal{B}_c(T)$ and $\mathcal{B}_o(T)$ be the σ -rings generated by the compact subsets and by the compact G_δ subsets of T , respectively. The members of $\mathcal{B}_c(T)$ are called σ -Borel sets of T since they are precisely the σ -bounded Borel sets of T . The members of $\mathcal{B}_o(T)$ are called the Baire sets of T . $M(T)$ denotes the dual of $C_o(T)$. Let X be a quasicomplete locally convex Hausdorff space. Suppose $u : C_o(T) \rightarrow X$ is a continuous linear operator. Using the Baire and σ -Borel characterizations of weakly compact sets in $M(T)$ as given in a previous paper of the author's and combining the integration technique of Bartle, Dunford and Schwartz, we obtain 35 characterizations for the operator u to be weakly compact, several of which are new. The independent results on the regularity and on the regular Borel extendability of σ -additive X -valued Baire measures are deduced as an immediate consequence of these characterizations. Some other applications are also included.

1. INTRODUCTION

For a locally compact Hausdorff space T , let $C_o(T)$ be the Banach space of all continuous complex functions vanishing at infinity in T , endowed with the supremum norm. Let $\mathcal{B}(T)$ be the σ -algebra of Borel sets in T , which is the σ -algebra generated by the class of all open sets in T . The dual of $C_o(T)$ is the Banach space $M(T)$ of all bounded complex Radon measures on T , and by Theorem 5.3 of [Pa2] it is isometrically isomorphic to the Banach space of all Borel regular complex measures μ on $\mathcal{B}(T)$, endowed with the norm $\|\mu\| = |\mu|(T)$, where $|\mu|$ is the variation of μ in $\mathcal{B}(T)$. Hence, we identify $M(T)$ with the Banach space of all Borel regular complex measures on $\mathcal{B}(T)$. Let X be a quasicomplete locally convex Hausdorff space (briefly, a quasicomplete lcHs) and let $u : C_o(T) \rightarrow X$ be a continuous linear map. Grothendieck gave in [G] some necessary and sufficient conditions for u to be weakly compact.

Received by the editors November 17, 1995.

1991 *Mathematics Subject Classification*. Primary 47B38, 46G10; Secondary 28B05.

Key words and phrases. Weakly compact operators, representing measure, vector measure, quasicomplete locally compact Hausdorff space, Borel (resp. σ -Borel, Baire) regularity, inner regularity and outer regularity.

Supported by the C.D.C.H.T. project C-586 of the Universidad de los Andes, Mérida, and by the international cooperation project between CONICIT-Venezuela and CNR-Italy.

He studied in [G] some topological and range properties of the adjoint u^* and the biadjoint u^{**} of the continuous linear operator $u : C_o(T) \rightarrow X$, characterized weakly compact subsets of $M(T)$ and proved some deep results such as Theorems 1 and 3 and Proposition 11 of [G], to obtain the characterization theorem ([G], Theorem 6) for weakly compact operators u on $C_o(T)$.

Most of the results obtained in Sections 1.1, 1.2, 2.1 and 3.1 of [G] play a key role in the proof of the said characterization theorem. Moreover, the major part of the results proved in [G] are given only for the space $C(K)$ with K compact Hausdorff, and it is remarked that the results hold also for $C_o(T)$ with T locally compact and Hausdorff. Later, the results of Grothendieck [G] were proved in detail for the locally compact case in Sections 4.21, 4.22 and 9.1–9.4 in Edwards [E]. (In this context see Remarks 3 and 4 below.)

Recently, in [Pa4] we obtained several characterizations of weakly compact sets in $M(T)$ in terms of the Baire and σ -Borel restrictions of the members of the set in question. Using these characterizations and those of Theorem 2 of [G] and combining the integration technique of Bartle-Dunford-Schwartz [BDS], we obtain 35 characterizations for a continuous quasicomplete lchS-valued operator on $C_o(T)$ to be weakly compact. Several of them are new. The advantage of the present study is that as in [BDS] the strict Dunford-Pettis property of $C_o(T)$ is not needed and, in fact, it is deduced from these characterization theorems. Also the use of other deep results such as Theorem 3 and Proposition 11 of [G] is dispensed with. Moreover, all the arguments are given directly for the locally compact space case, without reducing to the compact or compact metrizable case as is done in [E, G].

Further, the isolated results on σ -additive vector valued Baire measures as in [DK, DL, K] are deduced as consequences of the present study. Also included are new proofs of Theorem 5 of [Pe] and of Theorem 5.3 (bounded Radon measure case) of [Th], avoiding the technique of reduction to the metrizable compact case.

2. PRELIMINARIES

In this section we fix notation and terminology and also give some definitions and results from [Di], [E], [G], [Ha] and [Pa4].

Let T be a locally compact Hausdorff space and let $C_o(T)$ be the Banach space of all complex continuous functions vanishing at infinity in T , endowed with the supremum norm $\|\cdot\|_T$. Let $\mathcal{B}(T)$ be the σ -algebra of Borel sets in T , which is the σ -algebra generated by the class of all open sets in T . Then the dual of $C_o(T)$ is the Banach space $M(T)$ of all bounded complex Radon measures on T , and by Theorem 5.3 of [Pa2] it is isometrically isomorphic to the Banach space of all Borel regular complex measures μ on $\mathcal{B}(T)$ endowed with the norm $\|\mu\| = \text{var}(\mu, T)$, where the variation of μ is taken with respect to $\mathcal{B}(T)$. We denote by $\mathcal{C}(T)$ (resp. $\mathcal{C}_o(T)$) the class of all compact subsets (resp. compact G_δ subsets) of T . $\mathcal{B}_c(T)$ is the σ -ring generated by the class $\mathcal{C}(T)$, and its members are called σ -Borel sets of T since they are precisely the σ -bounded Borel sets of T . $\mathcal{B}_o(T)$ is the σ -ring generated by the class $\mathcal{C}_o(T)$, and its members are called the Baire sets of T .

We recall the following lemma from [Pa4] before proceeding with some definitions.

Lemma 1. *For $\mu \in M(T)$, let $|\mu|(\cdot) = \text{var}(\mu, (\cdot))$ in $\mathcal{B}(T)$. Then*

$$|\mu|_{\mathcal{B}_o(T)}(\cdot) = \text{var}(\mu|_{\mathcal{B}_o(T)}, (\cdot)) \text{ and } |\mu|_{\mathcal{B}_c(T)}(\cdot) = \text{var}(\mu|_{\mathcal{B}_c(T)}, (\cdot)).$$

Notation 1. For $\mu \in M(T)$, let $|\mu|(\cdot) = \text{var}(\mu, (\cdot))$ on $\mathcal{B}(T)$.

In the light of Lemma 1, the variations used in the following definition are unambiguously defined.

Definition 1. Let \mathcal{S} be a σ -ring of sets in T such that $\mathcal{C}(T) \subset \mathcal{S}$ or $\mathcal{C}_o(T) \subset \mathcal{S}$. A complex measure μ on \mathcal{S} is said to be \mathcal{S} -regular if, given $E \in \mathcal{S}$ and $\varepsilon > 0$, there exist a compact set $K \in \mathcal{S}$ and an open set $U \in \mathcal{S}$ with $K \subset E \subset U$ such that $|\mu(B)| < \varepsilon$ for every $B \in \mathcal{S}$ with $B \subset U \setminus K$. When $\mathcal{S} = \mathcal{B}(T)$ (resp. $\mathcal{B}_c(T)$, $\mathcal{B}_o(T)$), we use the terminology *Borel* (resp. *σ -Borel*, *Baire*) *regularity* in place of \mathcal{S} -regularity. Let A be a subset of $M(T)$. We say that A is *uniformly Baire inner regular* (resp. *Baire regular*) in a set $E \in \mathcal{B}_o(T)$ if, given $\varepsilon > 0$, there exists a compact $K \in \mathcal{B}_o(T)$ with $K \subset E$ (resp. and an open Baire set O in T with $K \subset E \subset O$) such that $\sup_{\mu \in A} |\mu|(E \setminus K) < \varepsilon$ (resp. such that $\sup_{\mu \in A} |\mu|(O \setminus K) < \varepsilon$). If A is uniformly Baire inner regular (resp. Baire regular) in each Baire set, then A is said to be *uniformly Baire inner regular* (resp. *Baire regular*). Similarly, the *uniform Borel* (resp. *σ -Borel*) *regularity* and *inner regularity* of A and those of A in a Borel (resp. σ -Borel) set E are defined.

By Theorem 51.D of [Ha], a compact Baire set in T is a G_δ .

It is well known that every complex Baire measure μ_o is Baire regular and that it has a unique extension μ to $\mathcal{B}(T)$ (resp. μ_c to $\mathcal{B}_c(T)$) such that μ is a Borel (resp. μ_c is a σ -Borel) regular complex measure. Moreover, $\mu|_{\mathcal{B}_c(T)} = \mu_c$. (See, for example, Theorem 2.4 of [Pa2].)

The following proposition which is a combination of Theorems 1 and 2 of [Pa4], plays a key role in Section 5.

Proposition 1. *Let A be a bounded set in $M(T)$. Then the following statements are equivalent:*

- (i) A is relatively weakly compact.
- (ii) For each disjoint sequence (U_i) of open Baire sets (resp. (ii)' open sets, (ii)'' σ -Borel open sets) in T , $\lim_i \mu(U_i) = 0$ uniformly in $\mu \in A$.
- (iii) (a) A is uniformly Baire (resp. (a)' Borel, (a)'' σ -Borel) inner regular in each open Baire (resp. open, σ -Borel open) set U in T .
 (b) For each $\varepsilon > 0$, there exists a compact $K \in \mathcal{B}_o(T)$ (resp. (b)'' $K \in \mathcal{C}(T)$) such that

$$\sup_{\mu \in A} |\mu|(T \setminus K) < \varepsilon.$$

(Note that (i) \Leftrightarrow (iii)(a)'.)

- (iv) A is uniformly Baire (resp. (iv)' Borel, (iv)'' σ -Borel) inner regular.
- (v) $A|_{\mathcal{B}_o(T)}$ (resp. (v)' $A|_{\mathcal{B}(T)}$, (v)'' $A|_{\mathcal{B}_c(T)}$) is uniformly σ -additive.
- (vi) A is uniformly Baire (resp. (vi)' Borel, (vi)'' σ -Borel) regular.

The following result is also used in Section 5.

Proposition 2. $\mathcal{B}_o(T)$ is also the σ -ring generated by the class of all relatively compact open Baire sets. Each open Baire set U in T is a countable union of compact G_δ s in T .

For a proof of the above proposition the reader can refer to §14, Chapter III of [Di].

A vector measure is an additive set function defined on a ring of sets with values in a lchS. In the sequel X denotes a lchS with topology τ .

The *strong topology* $\beta(X^*, X)$ of X^* is the locally convex topology induced by the seminorms $\{p_B : B \text{ bounded in } X\}$, where $p_B(x^*) = \sup_{x \in B} |x^*(x)|$. X^{**} denotes the dual of $(X^*, \beta(X^*, X))$ and is endowed with the *locally convex topology* τ_e of *uniform convergence in equicontinuous subsets* of X^* . Note that $(X^*, \beta(X^*, X))$ and (X^{**}, τ_e) are lchSs.

It is well known that the canonical injection $J : X \rightarrow X^{**}$ given by $\langle Jx, x^* \rangle = \langle x, x^* \rangle$ for all $x \in X$ and $x^* \in X^*$ is linear. On identifying X with $JX \subset X^{**}$, one has $\tau_e|_{JX} = \tau_e|_X = \tau$.

Definition 2. A linear mapping $u : C_o(T) \rightarrow X$ is called a *weakly compact operator* on $C_o(T)$ if $\{uf : \|f\|_T \leq 1\}$ is relatively weakly compact in X .

Let E and F be lchSs and let $u : E \rightarrow F$ be a continuous linear map. Then the adjoint u^* and the biadjoint u^{**} of u are well defined linear maps with $u^* : F^* \rightarrow E^*$ and $u^{**} : E^{**} \rightarrow F^{**}$.

Proposition 3. Let E and F be lchSs and let $u : E \rightarrow F$ be a continuous linear map. Then $u^* : (F^*, \sigma(F^*, F)) \rightarrow (E^*, \sigma(E^*, E))$ is continuous and $u^{**} : (E^{**}, \tau_e) \rightarrow (F^{**}, \tau_e)$ is continuous.

See the corollary to Proposition 1, §12, Chapter 3 of [Ho] and Proposition 8.7.2 of [E].

The following result (Corollary 9.3.2 of [E], which is essentially due to Lemma 1 of [G]) is indispensable in the study of weakly compact operators on $C_o(T)$ in Section 5.

Proposition 4. Let E and F be lchSs with F quasicomplete. If $u : E \rightarrow F$ is linear and continuous, then u maps bounded subsets of E into relatively weakly compact subsets of F if and only if $u^*(A)$ is relatively $\sigma(E^*, E^{**})$ -compact for each equicontinuous subset A of F^* .

3. INTEGRATION OF BOUNDED \mathcal{S} -MEASURABLE SCALAR FUNCTIONS

In this section we define the integral of a bounded \mathcal{S} -measurable scalar function with respect to an X -valued vector measure m defined on a σ -ring \mathcal{S} , where m is either σ -additive or bounded and X is a quasicomplete lchS. To this end, we introduce the following additional notation and terminology.

For each τ -continuous seminorm p on X , let $p(x) = \|x\|_p$, $x \in X$, and let $X_p = (X, \|\cdot\|_p)$ be the associated seminormed space. The completion of the quotient normed space $X_p/p^{-1}(0)$ is denoted by \tilde{X}_p . Let $\Pi_p : X_p \rightarrow X_p/p^{-1}(0) \subset \tilde{X}_p$ be the canonical quotient map.

Let \mathcal{S} be a σ -ring of subsets of a nonempty set Ω . Given a vector measure $m : \mathcal{S} \rightarrow X$, for each τ -continuous seminorm p on X let $m_p : \mathcal{S} \rightarrow \tilde{X}_p$ be given by $m_p(E) = \Pi_p \circ m(E)$ for $E \in \mathcal{S}$. Then m_p is a Banach space valued vector measure on \mathcal{S} . We define the p -semivariation $\|m\|_p$ of m by

$$\|m\|_p(E) = \|m_p\|(E) \text{ for } E \in \mathcal{S}$$

and

$$\|m\|_p(\Omega) = \|m_p\|(\Omega) = \sup_{E \in \mathcal{S}} \|m_p\|(E),$$

where $\|m_p\|$ is the semivariation of the vector measure $m_p : \mathcal{S} \rightarrow \tilde{X}_p$. When m is σ -additive, then m_p is a Banach space valued σ -additive vector measure, and hence, by a well known result on vector measures, $\|m\|_p(\Omega) = \|m_p\|(\Omega) < \infty$.

An X -valued vector measure m on a σ -ring \mathcal{S} of subsets of Ω is said to be *bounded* if for each continuous seminorm p on X , $\|m\|_p(\Omega) < \infty$.

For an \mathcal{S} -simple scalar function $s = \sum_{i=1}^r \lambda_i \chi_{E_i}$, $\lambda_i \neq 0$, $E_i \in \mathcal{S}$, $E_i \cap E_j = \emptyset$ for $i \neq j$, $i, j = 1, 2, \dots, r$, and for an lcHs-valued bounded vector measure m we define

$$\int_E s dm = \sum_{i=1}^r \lambda_i m(E \cap E_i)$$

for $E \in \mathcal{S}$. It is easy to verify that $\int_E s dm$ is well defined, and for each E fixed in \mathcal{S} it is linear on the normed space $S(\mathcal{S})$ of all \mathcal{S} -simple scalar functions endowed with the supremum norm $\|\cdot\|_\Omega$.

Proposition 5. *Let (s_n) and (s'_n) be sequences of \mathcal{S} -simple scalar functions, converging uniformly to a function f in Ω . Suppose $m : \mathcal{S} \rightarrow X$ is σ -additive or is a bounded vector measure, where X is a quasicomplete lcHs. Then:*

- (i) $(\int_E s_n dm)$ is uniformly Cauchy in X for $E \in \mathcal{S}$.
- (ii) $\lim_n \int_E s_n dm = \lim_n \int_E s'_n dm \in X$ for each $E \in \mathcal{S}$.

Proof. Let p be a τ -continuous seminorm on X and let $\varepsilon > 0$. Since $\|m_p\|(\Omega) < \infty$, we can choose n_o such that $\|s_n - s_\ell\|_\Omega < \frac{\varepsilon}{\|m_p\|(\Omega)}$ for $n, \ell \geq n_o$. Then

$$\left\| \int_E s_n dm - \int_E s_\ell dm \right\|_p = \left\| \int_E (s_n - s_\ell) dm_p \right\|_p \leq \|s_n - s_\ell\|_\Omega \|m_p\|(\Omega) < \varepsilon$$

for $n, \ell \geq n_o$ and for all $E \in \mathcal{S}$. Hence (i) holds.

As X is sequentially complete, by (i) there exist vectors x_E, x'_E in X such that $\lim_n \int_E s_n dm = x_E$ and $\lim_n \int_E s'_n dm = x'_E$ for $E \in \mathcal{S}$. Then

$$\begin{aligned} \|x_E - x'_E\|_p &\leq \left\| \int_E s_n dm - x_E \right\|_p + \left\| \int_E s_n dm - \int_E s'_n dm \right\|_p \\ &\quad + \left\| \int_E s'_n dm - x'_E \right\|_p \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, since

$$\left\| \int_E s_n dm - \int_E s'_n dm \right\|_p \leq \|s_n - s'_n\|_\Omega \|m_p\|(\Omega) \rightarrow 0$$

as $n \rightarrow \infty$. As the τ -continuous seminorm p is arbitrary, it follows that $x_E = x'_E$ for each $E \in \mathcal{S}$. Hence (ii) holds. \square

Definition 3. Let f be a bounded \mathcal{S} -measurable scalar function so that there exists a sequence (s_n) of \mathcal{S} -simple scalar functions such that $s_n \rightarrow f$ uniformly in Ω . If $m : \mathcal{S} \rightarrow X$ is σ -additive or is additive and bounded, and X is a quasicomplete lcHs, then we say that f is *m-integrable* and define

$$\int_E f dm = \lim_n \int_E s_n dm$$

for each $E \in \mathcal{S}$.

In the light of Proposition 5, for a bounded \mathcal{S} -measurable scalar function f , $\int_E f dm$ is well defined. Since $N(f) = \{w \in \Omega : f(w) \neq 0\} \in \mathcal{S}$, we then define

$$\int_{\Omega} f dm = \int_{N(f)} f dm.$$

The following proposition is immediate from Definition 3.

Proposition 6. *Let X be a quasicomplete lcHs. If f and g are bounded \mathcal{S} -measurable scalar functions, α, β are scalars, $m : \mathcal{S} \rightarrow X$ is σ -additive or bounded and additive, and $E \in \mathcal{S}$, then the following hold:*

- (i) $\int_E (\alpha f + \beta g) dm = \alpha \int_E f dm + \beta \int_E g dm$.
- (ii) $\|\int_E f dm\|_p \leq \|f\|_E \|m_p\|(E) \leq \|f\|_{\Omega} \|m_p\|(\Omega) = \|f\|_{\Omega} \|m\|_p(\Omega)$ for each τ -continuous seminorm p on X .
- (iii) For each $x^* \in X^*$, $x^*(\int_E f dm) = \int_E f dx^*(m)$.

Proposition 7 (Lebesgue bounded convergence theorem). *Let X be a quasicomplete lcHs and let $m : \mathcal{S} \rightarrow X$ be σ -additive. If (f_n) is a bounded sequence of \mathcal{S} -measurable scalar functions with $\lim_n f_n(w) = f(w)$ for each $w \in \Omega$, then f is m -integrable and*

$$\int_E f dm = \lim_n \int_E f_n dm$$

for each $E \in \mathcal{S}$.

Proof. For each $E \in \mathcal{S}$, $\mathcal{S}_E = \mathcal{S} \cap E$ is a σ -algebra in E . Then for each τ -continuous seminorm p on X , by Theorem II.4.1 of [DU], we have

$$\|\int_E f_n dm - \int_E f dm\|_p = \|\int_E f_n dm_p - \int_E f dm_p\|_p \rightarrow 0$$

as $n \rightarrow \infty$. Hence the proposition holds. \square

Remark 1. Using the Egoroff-Lusin theorem for continuous submeasures, one can also show that the limit in the above proposition is uniform with respect to $E \in \mathcal{S}$. (See Theorem 1 of [Do].)

4. REPRESENTATION OF X -VALUED CONTINUOUS LINEAR TRANSFORMATIONS ON $C_o(T)$

The following result is analogous to Theorem VI.2.1 of [DU] for a lcHs-valued continuous linear map on $C_o(T)$, and plays a fundamental role in Sections 5 and 6.

Theorem 1. *Let X be an lcHs. Let $u : C_o(T) \rightarrow X$ be a continuous linear transformation. Then there exists an X^{**} -valued vector measure m on $\mathcal{B}(T)$ with the following properties:*

- (i) $x^*m \in M(T)$ for each $x^* \in X^*$, and consequently $m : \mathcal{B}(T) \rightarrow X^{**}$ is σ -additive in the $\sigma(X^{**}, X^*)$ -topology.
- (ii) The mapping $x^* \rightarrow x^*m$ of X^* into $M(T)$ is weak*-weak* continuous.
- (iii) $x^*uf = \int_T f dx^*m$ for each $f \in C_o(T)$ and $x^* \in X^*$.
- (iv) $\{m(E) : E \in \mathcal{B}(T)\}$ is τ_e -bounded in X^{**} .

Conversely, let $m : \mathcal{B}(T) \rightarrow X^{**}$ be a vector measure satisfying (i) and (ii). Then there exists a unique continuous linear transformation $u : C_o(T) \rightarrow X$ such that (iii) holds. Moreover, $m(E) = u^{**}(\chi_E)$ for $E \in \mathcal{B}(T)$ and m satisfies (iv).

Finally, the vector measure m satisfying (i)-(iii) is uniquely determined by the continuous linear transformation u and has τ_e -bounded range in X^{**} .

Proof. Note that $u^* : X^* \rightarrow M(T)$ and $u^{**} : C_o^{**}(T) \rightarrow X^{**}$ are linear. Since each bounded Borel measurable scalar function f on T can be identified with a unique element $\iota(f) \in C_o^{**}(T)$ such that $\|f\|_T = \|\iota(f)\|$, we shall identify $\{\chi_E : E \in \mathcal{B}(T)\}$ as a subset of $C_o^{**}(T)$. Let us define $m : \mathcal{B}(T) \rightarrow X^{**}$ by putting $m(E) = u^{**}(\chi_E)$ for $E \in \mathcal{B}(T)$. Then m is clearly additive. Moreover, $u^*x^* \in M(T)$ for each $x^* \in X^*$ and hence there exists a unique $\mu_{x^*} \in M(T)$ such that $u^*x^* = \mu_{x^*}$. Consequently,

$$\mu_{x^*}(E) = (u^*x^*)(E) = \langle u^*x^*, \chi_E \rangle = \langle x^*, u^{**}(\chi_E) \rangle = \langle x^*, m(E) \rangle = (x^*m)(E)$$

for each $E \in \mathcal{B}(T)$. Hence $\mu_{x^*} = u^*x^* = x^*m$ for $x^* \in X^*$. Thus (i) holds.

By the first part of Proposition 3,

$$u^* : (X^*, \sigma(X^*, X)) \rightarrow (M(T), \sigma(M(T), C_o(T)))$$

is continuous, whence (ii) follows.

Since

$$x^*(uf) = \langle uf, x^* \rangle = \langle f, u^*x^* \rangle = \langle f, x^*m \rangle = \int_T f d(x^*m)$$

for each $f \in C_o(T)$ and $x^* \in X^*$, (iii) holds.

By the second part of Proposition 3 and by Theorem 1.32 of Rudin [R], $\{m(E) : E \in \mathcal{B}(T)\} = \{u^{**}(\chi_E) : E \in \mathcal{B}(T)\}$ is τ_e -bounded in X^{**} , and hence (iv) holds.

Conversely, let $m : \mathcal{B}(T) \rightarrow X^{**}$ be a vector measure satisfying (i) and (ii). Let $x^* \in X^*$. Then by (i) there exists $\mu_{x^*} \in M(T)$ such that $x^*m = \mu_{x^*}$, and by (ii) the mapping $V : X^* \rightarrow M(T)$ given by $Vx^* = \mu_{x^*}$ is linear and weak*-weak* continuous. For $f \in C_o(T)$, let $V_f : X^* \rightarrow \mathbb{C}$ be given by $V_f(x^*) = \langle f, Vx^* \rangle = \int_T f d\mu_{x^*}$. Then by the weak*-weak* continuity of V it follows that V_f is a $\sigma(X^*, X)$ -continuous linear functional on X^* for each $f \in C_o(T)$. Consequently, there exists a unique vector $x_f \in X$ such that $V_fx^* = x^*(x_f)$ for each $x^* \in X^*$. Let $uf = x_f$, $f \in C_o(T)$. Clearly, $u : C_o(T) \rightarrow X$ is linear. Moreover,

$$\begin{aligned} \sup_{\|f\|_T \leq 1} |x^*uf| &= \sup_{\|f\|_T \leq 1} |x^*x_f| = \sup_{\|f\|_T \leq 1} |V_fx^*| \\ &= \sup_{\|f\|_T \leq 1} \left| \int_T f d(x^*m) \right| \leq |x^*m|(T) < \infty \end{aligned}$$

for each $x^* \in X^*$. Consequently, $\{uf : \|f\|_T \leq 1\}$ is weakly bounded in X and hence τ -bounded by Theorem 3.18 of Rudin [R]. Then by Theorem 1.32 of Rudin [R], u is continuous. Moreover,

$$(4.1) \quad x^*uf = x^*x_f = V_fx^* = \int_T f d(x^*m)$$

for each $f \in C_o(T)$ and $x^* \in X^*$. Thus the continuous linear transformation u satisfies (iii).

If there exists another continuous linear transformation $v : C_o(T) \rightarrow X$ such that $x^*vf = \int_T f d(x^*m)$ for each $x^* \in X^*$ and $f \in C_o(T)$, then by (4.1) and by the Hahn-Banach theorem we have $v = u$.

Let m_1 be the vector measure defined on $\mathcal{B}(T)$ by $m_1(E) = u^{**}(\chi_E)$. Then as shown in the proof of the first part we have $u^*x^* = x^*m_1$ for each $x^* \in X^*$,

$x^*uf = \int_T fd(x^*m_1)$ for $f \in C_o(T)$ and $x^* \in X^*$, and m_1 has τ_e -bounded range in X^{**} . Then by (4.1) we have

$$x^*uf = \int_T fd(x^*m_1) = \int_T fd(x^*m)$$

for $f \in C_o(T)$. Since x^*m, x^*m_1 are regular complex Borel measures on T , by the uniqueness part of the Riesz representation theorem it follows that $x^*m = x^*m_1$ on $\mathcal{B}(T)$ for each $x^* \in X^*$, and hence $m_1 = m$. Consequently, m has τ_e -bounded range. \square

Definition 4. Given a continuous linear transformation $u : C_o(T) \rightarrow X$, the unique X^{**} -valued vector measure m satisfying (i)-(iii) of Theorem 1 is called the *representing measure* of u .

5. CHARACTERIZATIONS OF X -VALUED WEAKLY COMPACT OPERATORS ON $C_o(T)$

Making use of the results of the earlier sections, we obtain 35 characterizations of a quasicomplete lcHs-valued weakly compact operator on $C_o(T)$. Among them are Gothendieck's characterizations from Remark 2 of [G], as well as several new ones. For convenience of exposition we give these characterizations through eight theorems, instead of all in a single theorem. The equivalent statements given in these theorems are numbered successively, and they will be referred to as (viii), (xii), etc. without mentioning the theorem in which they appear.

Motivated by Definition 3.2 of [Pa1], we give the following notions of regularity for a vector measure.

Definition 5. Let \mathcal{R} be a ring of sets in T with $\mathcal{R} \supset \mathcal{C}(T)$ or $\mathcal{R} \supset C_o(T)$. Let $m : \mathcal{R} \rightarrow X$ be a vector measure. Then m is said to be \mathcal{R} -regular or simply regular (resp. \mathcal{R} -outer regular or simply outer regular, \mathcal{R} -inner regular or simply inner regular) in $E \in \mathcal{R}$ if, given $\varepsilon > 0$ and a τ -continuous seminorm p on X , there exist a compact set $K \in \mathcal{R}$ and an open set $U \in \mathcal{R}$ with $K \subset E \subset U$ (resp. there exists an open set $U \in \mathcal{R}$ with $E \subset U$, there exists a compact set $K \in \mathcal{R}$ with $K \subset E$) such that for each $B \in \mathcal{R}$ with $B \subset U \setminus K$ (resp. with $B \subset U \setminus E$, with $B \subset E \setminus K$) we have $p(m(B)) < \varepsilon$. Even though T does not belong to \mathcal{R} , one can define \mathcal{R} -inner regularity of m in T on similar lines. The vector measure m is said to be \mathcal{R} -regular or simply regular (resp. \mathcal{R} -outer regular or simply outer regular, \mathcal{R} -inner regular or simply inner regular) if it is so in each $E \in \mathcal{R}$. When $\mathcal{R} = \mathcal{B}(T)$ (resp. $\mathcal{B}_c(T), \mathcal{B}_o(T)$), we use the terminology *Borel* (resp. σ -Borel, Baire) regularity or *outer regularity* or *inner regularity*.

Remark 2. In Definition 5 one can restrict to any family of τ -continuous seminorms on X which induces the topology τ in X .

Notation 2. For a set E in X^* , $p_E(x^{**}) = \sup_{x^* \in E} |x^{**}(x^*)|$, $x^{**} \in X^{**}$.

Lemma 2. Let X be a quasicomplete lcHs and let $u : C_o(T) \rightarrow X$ be a continuous linear map. Then u^*E is bounded in $M(T)$ for each equicontinuous set E in X^* .

Proof. Let m be the representing measure of u . Then from the proof of Theorem 1(i) we have $u^*E = \{x^* \circ m : x^* \in E\}$. As m has τ_e -bounded range by Theorem

1(iv), it follows that

$$\sup_{x^* \in E} |x^* \circ m|(T) \leq 4 \sup_{x^* \in E, A \in \mathcal{B}(T)} |(x^* \circ m)(A)| = 4 \sup_{A \in \mathcal{B}(T)} p_E(m(A)) < \infty.$$

Thus u^*E is bounded in $M(T)$. \square

Theorem 2. Let $u : C_o(T) \rightarrow X$ be a continuous linear transformation, where X is a quasicomplete lcHs, and let $m : \mathcal{B}(T) \rightarrow X^{**}$ be the representing measure of u . Then the following statements are equivalent:

- (i) u is weakly compact.
- (ii) The range of m is contained in X .
- (iii) m is σ -additive in the topology τ_e of X^{**} .

Proof. (i) \Rightarrow (ii). By the equivalence of (1) and (2) of Lemma 1 of Grothendieck [G], u is weakly compact if and only if $u^{**}(C_o^{**}(T)) \subset X$. Since $m(E) = u^{**}(\chi_E)$ for $E \in \mathcal{B}(T)$, (i) \Rightarrow (ii). (Note that the quasicompleteness of X is not needed for this implication.)

(ii) \Rightarrow (iii). By Theorem 1(i) and the hypothesis (ii), the vector measure m is σ -additive in $\sigma(X, X^*)$ -topology. Since $\tau_e|_X = \tau$, now (iii) holds by the Orlicz-Pettis theorem (see, for example, [M]).

(iii) \Rightarrow (i). Let (U_n) be a disjoint sequence of open sets in T and let E be an equicontinuous subset of X^* . If $U = \bigcup_{n=1}^{\infty} U_n$, then (iii) implies that $\|m(U_n)\|_{p_E} \rightarrow 0$ as $n \rightarrow \infty$. In other words, $\lim_n x^* \circ m(U_n) = 0$ uniformly in $x^* \in E$. Since u^*E is bounded in $M(T)$ by Lemma 2, and since $u^*x^* = x^* \circ m$, by the equivalence of (i) and (ii)' of Proposition 1 it follows that u^*E is relatively weakly compact in $M(T)$. Then by Proposition 4, u is weakly compact and hence (i) holds. \square

Theorem 3. Let X , u and m be as in Theorem 2. Then u is weakly compact if and only if any one of the following conditions holds:

- (iv) $m(U) \in X$ for all open sets U in T .
- (v) $m(F) \in X$ for all closed sets F in T .
- (vi) $m(U) \in X$ for all open sets U in T which are σ -Borel.
- (vii) $m(U) \in X$ for all open Baire sets U in T .
- (viii) $m(U) \in X$ for all open sets U in T which are σ -compact in T .
- (ix) $m(F) \in X$ for all closed sets F in T which are G_δ .
- (x) $m(U) \in X$ for all open sets U in T which are a countable union of closed sets in T .
- (xi) For each increasing sequence $(f_n)_1^\infty \subset C_0(T)$, with $0 \leq f_n \leq 1$, (uf_n) converges weakly in X .

Proof. Obviously (iv) \Rightarrow (v) \Rightarrow (ix); (iv) \Rightarrow (vi) \Rightarrow (vii); (iv) \Rightarrow (viii); (iv) \Rightarrow (x) and (x) \Rightarrow (viii). Moreover, (viii) \Rightarrow (vii) by Proposition 2 and (i) \Rightarrow (iv) by Theorem 2.

(vii) \Rightarrow (i). Let (U_n) be a disjoint sequence of open Baire sets in T and let $U = \bigcup_{n=1}^{\infty} U_n$. Then by Theorem 1(i), the hypothesis (vii) and the Orlicz-Pettis theorem, $m(U) = \sum_{n=1}^{\infty} m(U_n)$ in the topology τ of X . Let E be an equicontinuous set in X^* . Since $\tau_e|_X = \tau$, it follows that $\|m(U_n)\|_{p_E} \rightarrow 0$. Then the argument in the proof of (iii) \Rightarrow (i) of Theorem 2 holds here verbatim except that we have to invoke the equivalence of (i) and (ii) of Proposition 1 instead of that of (i) and (ii)' of Proposition 1. Hence u is weakly compact.

(ix) \Rightarrow (vii). Let U be an open Baire set in T . Then by Proposition 2, U is of the form $U = \bigcup_{n=1}^{\infty} K_n$ with $K_n \in \mathcal{C}_o(T)$ for each n . Then $T \setminus U = \bigcap_{n=1}^{\infty} (T \setminus K_n)$

is a G_δ . Thus (ix) implies that $m(T)$ and $m(T \setminus U)$ belong to X . Then $m(U) = m(T) - m(T \setminus U) \in X$, and hence (vii) holds.

(vii) \Rightarrow (viii) since (iv) \Leftrightarrow (vii) and (iv) \Rightarrow (viii).

(viii) \Rightarrow (x) since (viii) \Rightarrow (vii) \Leftrightarrow (iv), and (iv) \Rightarrow (x).

(i) \Rightarrow (xi). Let $(f_n)_1^\infty$ be an increasing sequence in $C_o(T)$, with $0 \leq f_n \leq 1$. Clearly, $\lim_n f_n(t) = f(t)$ exists in $[0, 1]$ for each $t \in T$, and moreover, f is Borel measurable. Then by Proposition 7, Theorem 2 and the fact that $\tau_e|_X = \tau$ we have

$$\lim_n \int_T f_n dm = \int_T f dm \in X$$

in the topology τ of X . Consequently, by Theorem 1(iii) and Proposition 6(iii)

$$\lim_n x^* u f_n = \lim_n \int_T f_n d(x^* m) = x^* \left(\int_T f dm \right)$$

for each $x^* \in X^*$. Hence (xi) holds.

(xi) \Rightarrow (vii). Let U be an open Baire set in T . Then by Proposition 2, U is the union of an increasing sequence (K_n) of compact G_δ subsets of T . Then by Urysohn's lemma there exists an increasing sequence $(g_n)_1^\infty$ of nonnegative continuous functions with compact support such that $g_n \nearrow \chi_U$. Therefore, by hypothesis there exists a vector $x_o \in X$ such that $x^* u g_n \rightarrow x^*(x_o)$ for each $x^* \in X^*$. On the other hand, by Theorem 1(iii) and by the Lebesgue dominated convergence theorem we have

$$\lim_n x^* u g_n = \lim_n \int_T g_n d(x^* m) = \int_T \chi_U d(x^* m) = (x^* m)(U)$$

for each $x^* \in X^*$. Thus $x^*(m(U) - x_o) = 0$ for each $x^* \in X^*$. Since $x_o \in X$ and $m(U) \in X^{**}$, it follows that $m(U) = x_o \in X$. Hence (vii) holds. \square

Remark 3. An alternative proof of (i) \Rightarrow (xi) without using Proposition 7 is given in Grothendieck [G]. Since (f_n) is weakly Cauchy, u is weakly compact and $C_o(T)$ has the strict Dunford-Pettis property, it follows that $(u f_n)$ is Cauchy in the quasicomplete lchS X and hence $(u f_n)$ converges in the topology τ of X . Thus, in particular, it is weakly convergent. Our proof does not use the strict Dunford-Pettis property of $C_o(T)$, which will later be deduced in Section 6 (see Theorem 11) as a consequence of Theorems 1 and 2 and Proposition 7. In this connection we would also like to point out that the argument in the proof of (1) implies (3) of Theorem 9.4.10 of Edwards [E] is incorrect and has to be replaced as above, invoking the strict Dunford-Pettis property of $C_o(T)$.

Remark 4. When T is compact and Hausdorff, Grothendieck proved in [G] the equivalence of the statements (i), (v), (ix) and (xi) (see Theorem 6 of [G]). Then he commented in Remark 2 on p.161 of [G] that the theorem holds also for locally compact Hausdorff spaces, but did not provide details. Later, Edwards provided details for the general case in the proof of Theorem 9.4.10 of [E], but unfortunately his proof of (3) \Rightarrow (2 bis) is incorrect. However, Theorems 2 and 3 above establish its validity in the general case. Moreover, even for the compact case, Grothendieck's proof needs his deep results such as Theorem 3 and Proposition 11 of [G]. On the other hand, our proof given here is not only more elementary but also is applicable to the general locally compact case.

Theorem 4. Let X , u and m be as in Theorem 2. Let $m_c = m|_{\mathcal{B}_c(T)}$ and $m_o = m|_{\mathcal{B}_o(T)}$. Then u is weakly compact if and only if any one of the following equivalent statements holds:

- (xii) m_c is σ -additive in the topology τ_e of X^{**} .
- (xiii) m_o is σ -additive in the topology τ_e of X^{**} .
- (xiv) m_c has range in X .
- (xv) m_o has range in X .

Proof. In the light of Theorem 2, it suffices to show that (xiii) \Rightarrow (i) and (xv) \Rightarrow (xiii).

(xiii) \Rightarrow (i). Let (U_n) be a disjoint sequence of open Baire sets in T with $U = \bigcup_1^\infty U_n$. Then, by hypothesis (xiii), $m_o(U) = \sum_1^\infty m_o(U_n)$ in the topology τ_e of X^{**} , and hence $\lim_n m_o(U_n) = 0$ in τ_e of X^{**} . Thus, for each equicontinuous set E in X^* , $\|m_o(U_n)\|_{p_E} \rightarrow 0$ as $n \rightarrow \infty$. In other words, $\lim_n (x^* \circ m_o)(U_n) = 0$ uniformly in $x^* \in E$. Since $u^*x^* = x^* \circ m$ and since u^*E is bounded in $M(T)$ by Lemma 2, by the equivalence of (i) and (ii) of Proposition 1 it follows that u^*E is relatively weakly compact in $M(T)$. Now by Proposition 4 we conclude that u is weakly compact.

(xv) \Rightarrow (xiii). Since m_o is σ -additive in the $\sigma(X^{**}, X^*)$ -topology by Theorem 1(i) and since m_o has range in X , by the Orlicz-Pettis theorem m_o is σ -additive in the topology τ of X . Since $\tau_e|_X = \tau$, the statement (xiii) holds. \square

Let us recall that a vector measure ν defined on a ring of sets \mathcal{R} with values in a lchS X is said to be *strongly additive* if for each disjoint sequence $(E_n) \subset \mathcal{R}$ we have $\lim_n \nu(E_n) = 0$. Since in the proof of (xiii) \Rightarrow (i) we have used only the fact that $\lim_n m_o(U_n) = 0$ for a disjoint sequence of open Baire sets (U_n) in T , and since σ -additivity in a σ -ring implies strong additivity, the following theorem is immediate.

Theorem 5. Let X, u, m, m_c and m_o be as in Theorem 4. Then u is weakly compact if and only if any one of the following conditions holds:

- (xvi) m is strongly additive in the topology τ_e of X^{**} .
- (xvii) m_c is strongly additive in the topology τ_e of X^{**} .
- (xviii) m_o is strongly additive in the topology τ_e of X^{**} .

Theorem 6. Let X, u and m be as in Theorem 2. Then u is weakly compact if and only if any one of the following conditions holds, where the regularities referred to are with respect to the topology τ_e of X^{**} .

- (xix) m is Borel regular.
- (xx) m is Borel inner regular.
- (xxi) m is Borel inner regular in each open set in T .
- (xxii) m is Borel outer regular in each compact set in T and Borel inner regular in the set T .

Proof. (i) \Rightarrow (xix). Let $A \in \mathcal{B}(T)$, $\varepsilon > 0$, and E an equicontinuous set in X^* . It suffices to show that there exist a compact K and an open set U in T with $K \subset A \subset U$ such that $p_E(m(B)) < \varepsilon$ for all $B \in \mathcal{B}(T)$ with $B \subset U \setminus K$. Since u is weakly compact, by Proposition 4 u^*E is relatively weakly compact in $M(T)$, and hence, by the equivalence of (i) and (vi)' of Proposition 1 and by the fact that $u^*x^* = x^* \circ m$, the statement (xix) holds.

Obviously, (xix) \Rightarrow (xx) \Rightarrow (xxi).

(xxi) \Rightarrow (i). Let U be an open set in T and let $\varepsilon > 0$. Given an equicontinuous set E in X^* , by hypothesis there exists a compact set $K \subset U$ such that $p_E(m(B)) < \varepsilon$ for all $B \in \mathcal{B}(T)$ with $B \subset U \setminus K$. Thus $\sup_{x^* \in E} |(x^* \circ m)(B)| < \varepsilon$ for all such B . Since $x^* \circ m = u^*x^*$ and since u^*E is bounded in $M(T)$ by Lemma 2, by the equivalence of (i) and (iii)(a)' of Proposition 1 it follows that u^*E is relatively weakly compact in $M(T)$. Then u is weakly compact by Proposition 4.

(xix) \Rightarrow (xxii). Obvious.

(xxii) \Rightarrow (i). Let K be compact in T and $\varepsilon > 0$. By hypothesis, given an equicontinuous set E in X^* , there exists an open set U with $K \subset U$ such that $p_E(m(B)) < \varepsilon$ for all $B \in \mathcal{B}(T)$ with $B \subset U \setminus K$. Then by Lemma 2 u^*E is bounded, and $\sup_{x^* \in E} |(x^* \circ m)(B)| < \varepsilon$ for all such B . The Borel inner regularity of m in T implies as in the proof of (xxi) \Rightarrow (i) that there exists a compact set K in T such that $\sup_{\mu \in u^*E} |\mu(B)| < \varepsilon$ for all $B \in \mathcal{B}(T)$ with $B \subset T \setminus K$. Thus by Theorem 4.22.1 of Edwards [E], u^*E is relatively weakly compact in $M(T)$. Now an appeal to Proposition 4 proves that u is weakly compact. \square

Theorem 7. *Let X , u , m and m_c be as in Theorem 4. Then u is weakly compact if and only if any one of the following conditions holds, where the regularities referred to are with respect to the topology τ_e of X^{**} .*

(xxiii) *The vector measure m_c is σ -Borel regular.*

(xxiv) *The vector measure m_c is σ -Borel inner regular.*

(xxv) *The vector measure m_c is σ -Borel inner regular in each σ -Borel open set U in T and in the set T .*

(xxvi) *The vector measure m_c is σ -Borel outer regular in each compact subset of T and is σ -Borel inner regular in the set T .*

Proof. Let E be an equicontinuous set in X^* and let $\varepsilon > 0$.

Following an argument similar to that in the proof of (i) \Rightarrow (xix) (resp. of (xxi) \Rightarrow (i) with obvious modifications) and appealing to the equivalence of (i) and (vi)" (resp. of (i) and (iii){(a)" and (b)"}) of Proposition 1, and to Proposition 4, one can show that (i) \Rightarrow (xxiii) (resp. (xxv) \Rightarrow (i)). Clearly, (xxiii) \Rightarrow (xxiv).

(xxiv) \Rightarrow (xxv). By Lemma 2 the set u^*E is bounded in $M(T)$. The hypothesis (xxiv) then implies that u^*E is uniformly σ -Borel inner regular, and consequently, by the equivalence of (iv)" and (iii){(a)" and (b)"}) of Proposition 1 the statement (xxv) holds.

(xxiii) \Rightarrow (xxvi). u^*E is bounded in $M(T)$ by Lemma 2, and the hypothesis implies that the set u^*E is uniformly σ -Borel regular. Thus by the equivalence of (i) and (vi)" of Proposition 1, the set u^*E is relatively weakly compact in $M(T)$. Then by Theorem 4.22.1 of Edwards [E] and by Theorem 50.D of [Ha] the statement (xxvi) holds.

(xxvi) \Rightarrow (i). By Lemma 2, u^*E is bounded in $M(T)$. Let K be a compact set in T . Then by hypothesis there exists a σ -Borel open set U in T such that $K \subset U$ and

$$\sup_{\mu \in u^*E} |\mu|(U \setminus K) < \varepsilon,$$

where the variation is taken with respect to $\mathcal{B}_c(T)$. Then by Lemma 1 it follows that (4)(a) of Theorem 4.22.1 of [E] is satisfied by u^*E in K . Again by hypothesis

there exists a compact C in T such that

$$(5.1) \quad \sup_{\mu \in u^*E} |\mu(B)| \leq \varepsilon$$

for all $B \in \mathcal{B}_c(T)$ with $B \subset T \setminus C$. Then, for each compact $K \subset T \setminus C$ and $x^* \in E$, by (5.1) and Lemma 1 we have

$$|x^* \circ m|(K) = \sup \left\{ \sum_{i=1}^n |(x^* \circ m)(B_i)| : (B_i)_{i=1}^n \subset \mathcal{B}_c(T), \right. \\ \left. B_i \cap B_j = \emptyset, i \neq j, \bigcup_{i=1}^n B_i = K \right\} \leq 4\varepsilon.$$

Since $|x^* \circ m|$ is Borel regular by Theorem 1(i), it follows that

$$|x^* \circ m|(T \setminus C) = \sup_{K \text{ compact}, K \subset T \setminus C} |x^* \circ m|(K) \leq 4\varepsilon.$$

Consequently, we have

$$\sup_{\mu \in u^*E} |\mu|(T \setminus C) \leq 4\varepsilon,$$

and thus (4)(b) of Theorem 4.22.1 of [E] is also satisfied by u^*E . Therefore, u^*E is relatively weakly compact in $M(T)$, and hence u is weakly compact by Proposition 4. \square

Remark 5. The above theorem can be compared with Corollary 4.9 of Panchapagesan [Pa3], where the vector measure is assumed to be σ -additive on $\mathcal{B}_c(T)$. Since the vector measure m is σ -additive only in the $\sigma(X^{**}, X^*)$ -topology of X^{**} , unlike the said corollary, the hypothesis of σ -Borel inner regularity of m_c in the set T has to be included in the statements (xxv) and (xxvi) for the validity of their equivalence with the σ -Borel regularity.

Theorem 8. *Let X, u, m and m_o be as in Theorem 4. Then u is weakly compact if and only if any one of the following equivalent conditions holds, where the regularities referred to are with respect to the topology τ_e of X^{**} .*

- (xxvii) *The vector measure m_o is Baire regular.*
- (xxviii) *The vector measure m_o is Baire inner regular.*
- (xxix) *The vector measure m_o is Baire inner regular in each open Baire set U in T and in the set T .*
- (xxx) *The vector measure m_o is Baire outer regular in each compact G_δ in T and Baire inner regular in the set T .*

Proof. The proof is similar to that of Theorem 7, except that we have to use Baire sets in place of σ -Borel sets in Proposition 1 and make use of Theorem 50.D of Halmos [Ha] in the last step to show that $\sup_{x^* \in E} |x^* \circ m|(T \setminus C) \leq 4\varepsilon$. The details are left to the reader. \square

Remark 6. It is well known that a σ -additive lCHs-valued Baire measure is Baire regular (see [DK], [DL]). Neither this result nor the Baire measure case of Corollary 4.9 of [Pa3] can hold here verbatim, as m is σ -additive only in $\sigma(X^{**}, X^*)$ -topology and hence the Baire regularity conditions have to be imposed on m_o for the validity of the above theorem.

Theorem 9. *Let X, u, m, m_c and m_o be as in Theorem 4. Then u is weakly compact if and only if any one of the following equivalent conditions holds:*

- (xxxi) *All bounded Borel measurable scalar functions on T are m -integrable and $\int_T f dm \in X$.*
- (xxxii) *All bounded $\mathcal{B}_c(T)$ -measurable scalar functions on T are m_c -integrable and $\int_T f dm_c \in X$.*
- (xxxiii) *All bounded Baire measurable scalar functions on T are m_o -integrable and $\int_T f dm_o \in X$.*
- (xxxiv) *All bounded scalar functions f belonging to the first Baire class in T are m_o -integrable and $\int_T f dm_o \in X$.*
- (xxxv) *$u^{**}f \in X$ for all bounded scalar functions f belonging to the first Baire class in T .*

Proof. By Theorem 1(iv), m has τ_e -bounded range in X^{**} . If u is weakly compact, then by Theorem 2, m has range in X . As X is quasicomplete, by the results of Section 3, (i) implies each one of the statements (xxxi)-(xxxiv). If (xxxi) (resp. (xxxii), (xxxiii)) holds, then for $A \in \mathcal{B}(T)$ (resp. $A \in \mathcal{B}_c(T)$, $A \in \mathcal{B}_o(T)$), $m(A) \in X$ (resp. $m_c(A) \in X$, $m_o(A) \in X$) and consequently, by (ii) (resp. by (xiv), by (xv)) u is weakly compact.

(xxxiv) \Rightarrow (vii). Let U be an open Baire set in T . Then by Proposition 2, U is the union of an increasing sequence of compact G_δ sets K_n in T . Consequently, by Urysohn's lemma there exists an increasing sequence (f_n) of nonnegative continuous functions with compact support in T such that $f_n(t) \rightarrow \chi_U(t)$ for each $t \in T$. Thus χ_U belongs to the first Baire class in T , and hence by hypothesis $m_o(U) \in X$. Thus (vii) holds.

(xxxiv) \Rightarrow (xxxv). Note that the bounded scalar functions of the first Baire class belong to $C_o^{**}(T)$. If (xxxiv) holds, then as (xxxiv) implies (vii) \Leftrightarrow (i), u is weakly compact and hence by (xv), $m_o(A) \in X$ for each $A \in \mathcal{B}_o(T)$. Moreover, by Theorem 1(iv), m_o has τ -bounded range in X . If f is a bounded Baire function, then f is m_o -integrable with $\int_T f dm_o \in X$, and hence for each $x^* \in X^*$ we have

$$\left\langle \int_T f dm_o, x^* \right\rangle = \int_T f dx^* \circ m_o = \int_T f du^* x^* = \langle f, u^* x^* \rangle = \langle u^{**}f, x^* \rangle.$$

Thus $u^{**}f \in X$. (Alternatively, one can also use Lemma 1 of [G].)

Consequently, (i) \Rightarrow (xxxiv) \Rightarrow (xxxv).

(xxxv) \Rightarrow (vii). Since $u^{**}(\chi_U) = m_o(U)$ for each open Baire set U in T and since χ_U belongs to the first Baire class as observed in the proof of (xxxiv) \Rightarrow (vii), it follows from the hypothesis that $m_o(U) \in X$ for all open Baire sets U in T . Hence (vii) holds. \square

Remark 7. In [Pa5] we will give further characterizations of weakly compact operators on $C_o(T)$ considered as weakly compact Radon operators.

6. APPLICATIONS

As an immediate consequence of the results of Section 5 we deduce the well known results on vector measures such as the regularity (resp. regular Borel extendability) of X -valued σ -additive Baire measures (resp. when X is quasicomplete), the strict Dunford-Pettis property of $C_o(T)$, and a theorem of Pelczyński [Pe] and of Thomas [Th] on weakly compact operators on $C_o(T)$. Also we give a generalization of Corollary VI.2.17 of [DU] to quasicomplete lcHs-valued operators on $C_o(T)$.

Theorem 10. *Let X be a lcHs and T a locally compact Hausdorff space. Then each X -valued σ -additive Baire measure μ_o is regular. If X is further quasicomplete, then there exists a unique X -valued σ -additive regular Borel (resp. regular σ -Borel) vector measure μ on $\mathcal{B}(T)$ (resp. μ_c on $\mathcal{B}_c(T)$) such that $\mu|_{\mathcal{B}_o(T)} = \mu_o$ (resp. $\mu_c|_{\mathcal{B}_o(T)} = \mu_o$). Besides, $\mu_c = \mu|_{\mathcal{B}_c(T)}$.*

Proof. Let \hat{X} be the completion of X . Since each $f \in C_o(T)$ is bounded and Baire measurable, by Definition 3 the function f is μ_o -integrable and $\int_T f d\mu_o \in \hat{X}$. Let

$$(6.1) \quad Vf = \int_T f d\mu_o, \quad f \in C_o(T).$$

Then $V : C_o(T) \rightarrow \hat{X}$ is linear. V is also continuous, since for each continuous seminorm p on \hat{X} , by Proposition 6(ii) we have

$$\|Vf\|_p \leq \|f\|_T \|\mu_o\|_p(T), \quad f \in C_o(T),$$

and $\|\mu_o\|_p(T) < \infty$. Let m be the representing measure of V . Then by Theorem 1, m is an X^{**} -valued vector measure on $\mathcal{B}(T)$ such that m is σ -additive in the $\sigma(X^{**}, X^*)$ -topology, $x^*m \in M(T)$ for each $x^* \in X^*$ and

$$(6.2) \quad x^*Vf = \int_T f dx^*m$$

for each $f \in C_o(T)$. Then by (6.1), (6.2) and Proposition 6(iii) we have

$$x^*Vf = \int_T f dx^*\mu_o = \int_T f d(x^*m)_o \quad \text{for } f \in C_o(T) \text{ and for } x^* \in X^*,$$

where $(x^*m)_o = (x^*m)|_{\mathcal{B}_o(T)}$. Then, as $(x^*m)_o$ and $x^*\mu_o$ are σ -additive scalar measures on $\mathcal{B}_o(T)$, by the uniqueness part of the Riesz representation theorem (Baire measure version) we conclude that $(x^*m)(A) = (x^*\mu_o)(A)$ for each $A \in \mathcal{B}_o(T)$. Since μ_o has range in \hat{X} and m has range in X^{**} , it follows that $m(A) = \mu_o(A)$ for all $A \in \mathcal{B}_o(T)$. Consequently, by Theorem 4 the operator V is weakly compact, and then by (xxvii) of Theorem 8, the Baire measure $\mu_o (= m_o = m|_{\mathcal{B}_o(T)})$ is regular (as a σ -additive Baire measure with values in X).

Moreover, by Theorem 2(ii) m and $m_c = m|_{\mathcal{B}_c(T)}$ have range in \hat{X} . Further, m is Borel regular and σ -additive by Theorems 6 and 2, while m_c is σ -Borel regular and σ -additive by Theorems 7 and 4. Now we shall assume that X is quasicomplete. Let $\mathcal{C}(E) = \{K : K \subset E, K \text{ compact}\}$ for $E \in \mathcal{B}(T)$. Let us define $K_1 \leq K_2$ for $K_1, K_2 \in \mathcal{C}(E)$ if $K_1 \subset K_2$. Similarly, let $\mathcal{U}(E) = \{O : E \subset O, O \text{ open}\}$ and let us define $O_1 \leq O_2$ for $O_1, O_2 \in \mathcal{U}(E)$ if $O_2 \subset O_1$. Let $\mathcal{U}_o(E) = \mathcal{B}_o(T) \cap \mathcal{U}(E)$ with the same partial ordering as in $\mathcal{U}(E)$. Then clearly $\mathcal{C}(E)$, $\mathcal{U}(E)$ and $\mathcal{U}_o(E)$ are directed sets. Given a balanced closed neighborhood W of 0 in \hat{X} and a compact set K in T , by the Borel regularity of m there exists $O_o \in \mathcal{U}(K)$ such that $m(K) - m(O) \in W$ for all $O \in \mathcal{U}(K)$ with $O \geq O_o$. Then in virtue of Theorem 50.D of Halmos [Ha], we have $m(K) - m(U) \in W$ for all open Baire sets U in T with $K \subset U \subset O_o$. Thus we have

$$(6.3) \quad m(K) = \lim_{U \in \mathcal{U}_o(K)} m(U) = \lim_{U \in \mathcal{U}_o(K)} m_o(U) \in X,$$

since the range of m_o is bounded in X and X is quasicomplete. Consequently, the inner Borel regularity of m in $E \in \mathcal{B}(T)$, (6.3), the τ -boundedness of $\{m(K) : K \in \mathcal{C}(T)\}$ and the quasicompleteness of X imply that $m(E) \in X$ for each $E \in \mathcal{B}(T)$. Therefore, the ranges of m and m_c are contained in X . This shows that μ_o has a

σ -additive regular Borel (resp. σ -Borel) extension m (resp. m_c) with values in X , and moreover, $m_c = m|_{\mathcal{B}_c(T)}$.

Finally, to prove the uniqueness of m (resp. of m_c), let m_1 (resp. m_2) be an X -valued Borel regular (resp. σ -Borel regular) σ -additive extension of μ_o . Then, as $Vf = \int_T f d\mu_o$, $f \in C_o(T)$, it follows that

$$x^* Vf = \int_T f dx^* \mu_o = \int_T f dx^* m_1 = \int_T f dx^* m_2 = \int_T f dx^* m = \int_T dx^* m_c$$

for $f \in C_o(T)$ and $x^* \in X^*$. Now by the uniqueness part of the Riesz representation theorem in the Borel version (resp. σ -Borel version), we conclude that $x^* m_1 = x^* m$ (resp. $x^* m_2 = x^* m_c$) for each $x^* \in X^*$. Then by the Hahn-Banach theorem, $m_1 = m$ (resp. $m_2 = m_c$). Define $\mu = m$ and $\mu_c = m_c$. \square

Remark 8. The regularity of a lchS-valued σ -additive Baire measure on a locally compact Hausdorff space was first proved by Dinculeanu and Kluvánek in [DK], and a direct proof was later given by Dinculeanu and Lewis in [DL]. Using the results of [DK], Kluvánek [K] proved that each Banach space valued σ -additive Baire measure on such spaces has a unique σ -additive Borel regular extension. Kluvánek's result can easily be extended to quasicomplete lchS-valued Baire measures. The idea of the present proof has its origin in the proof of Theorem 3.7 of Panchapagesan [Pa1]. Moreover, the last part of the above theorem generalizes Theorem 3.7 of [Pa1] and Theorem 2.4 of [Pa2] to vector valued σ -additive Baire measures.

Theorem 11. *$C_o(T)$ has the strict Dunford-Pettis property (briefly, (SDP)-property). That is, for each weakly compact operator $u : C_o(T) \rightarrow X$, X a quasicomplete lchS, u transforms each weak Cauchy sequence in $C_o(T)$ into a convergent sequence in X , and consequently, u transforms each weakly compact set in $C_o(T)$ into a relatively compact set in X if X is moreover metrizable.*

Proof. If $(f_n)_{n=1}^\infty$ is weakly Cauchy in $C_o(T)$, then (f_n) converges to a function f pointwise in T and $(f_n)_{n=1}^\infty$ is norm bounded. By Theorem 2 the representing measure m of u has range in X and is σ -additive in X . Consequently, by Definition 3, Theorem 1(iii), Proposition 7, and the Hahn-Banach theorem we have

$$\lim_n u f_n = \lim_n \int_T f_n dm = \int_T f dm \in X.$$

Hence $C_o(T)$ has the (SDP)-property. The second part is immediate from the first due to the Šmulian theorem (Theorem 8.12.1 of Edwards [E]). \square

Remark 9. The above result is due to Grothendieck [G], and was used in the proof of Theorem 6 of [G] (which treats only the compact case). The proofs of all the characterization theorems given in Section 5 are independent of the (SDP)-property of $C_o(T)$, and our proof of the above theorem is quite analogous to that of Corollary 3.3 of [BDS].

The following theorem is a generalization of Corollary VI.2.17 of [DU] to quasicomplete lchS-valued operators on $C_o(T)$.

Theorem 12. *Let $u : C_o(T) \rightarrow X$ be a continuous linear map and let X be a quasicomplete lchS. Then the following are equivalent:*

- (i) *u is unconditionally convergent; that is, u maps weakly unconditionally convergent Cauchy series into unconditionally convergent series.*

- (ii) u is weakly compact.
- (iii) u maps sequences that tend to zero weakly into sequences convergent to zero.
- (iv) u maps weak Cauchy sequences into convergent sequences.
- (v) If (f_n) is a bounded sequence in $C_o(T)$ with $f_n \cdot f_\ell = 0$ for $n \neq \ell$, then $\lim_n u(f_n) = 0$.

Proof. (i) \Rightarrow (ii). Let U be an open Baire set. Then by Proposition 2, U is of the form $U = \bigcup_{n=1}^{\infty} K_n$, with K_n compact G_δ for each n and $K_n \nearrow$. By Urysohn's lemma there exists an increasing sequence (g_n) of nonnegative continuous functions with compact support such that $g_n(t) \nearrow \chi_U(t)$ for each $t \in T$. Then by the Lebesgue dominated convergence theorem,

$$\mu(U) = \lim_n \int_T g_n d\mu = \lim_n \sum_{k=1}^n \left(\int_T g_{k+1} d\mu - \int_T g_k d\mu \right) + \int_T g_1 d\mu$$

for each $\mu \in M(T)$. Moreover,

$$\begin{aligned} \sum_{n=1}^{\infty} \left| \int_T (g_{n+1} - g_n) d\mu \right| &\leq \sum_{n=1}^{\infty} \int_T (g_{n+1} - g_n) d|\mu| \\ &= \lim_k \int_T g_k d|\mu| - \int_T g_1 d|\mu| \leq 2|\mu|(U) < \infty. \end{aligned}$$

Therefore the series $\sum_{n=1}^{\infty} (g_{n+1} - g_n) + g_1$ is an unconditionally weakly Cauchy series (see p.150 of [DU]), and hence by (i), there exists a vector $x \in X$ such that

$$\sum_{n=1}^{\infty} (u(g_{n+1}) - u(g_n)) + u(g_1) = \lim_n u(g_n) = x.$$

If m is the representing measure of u , then by Theorem 1(iii) and by the Lebesgue dominated convergence theorem we have

$$x^* m(U) = \lim_n x^* u(g_n) = x^* x$$

for each $x^* \in X^*$, and hence $m(U) = x \in X$. Consequently, u is weakly compact by Theorem 3(vii).

(ii) \Rightarrow (i) and (iv) by Theorem 11, and (iv) \Rightarrow (iii) obviously. (One can also apply the Orlicz-Pettis theorem to show (ii) \Rightarrow (i) as on p.150 of [DU].)

(iii) \Rightarrow (v). Such a norm bounded sequence (f_n) converges to zero pointwise in T and hence by the Lebesgue bounded convergence theorem is weakly convergent to zero. Thus (iii) implies (v).

(v) \Rightarrow (ii). If u is not weakly compact and if m is the representing measure of u , then by Propositions 1(ii) and 4 there would exist an equicontinuous set E in X^* , an $\varepsilon > 0$, a disjoint sequence (U_n) of open Baire sets in T and a sequence (x_n^*) in E such that $|(x_n^* \circ m)(U_n)| > \varepsilon$. By Propositions 2 and 7 and by Urysohn's lemma we can choose a continuous function f_n with compact support such that $0 \leq f_n \leq \chi_{U_n}$ and

$$(6.4) \quad \left| \int_T f_n d(x_n^* \circ m) \right| > \varepsilon$$

for each n . Then clearly (f_n) satisfies the hypothesis of (v), and consequently $\lim_n u(f_n) = 0$. Thus there exists n_o such that $p_E(u(f_n)) = \sup_{x^* \in E} |x^*(u(f_n))| < \varepsilon$ for all $n \geq n_o$. Then by Theorem 1 (iii), this means that $\sup_{x^* \in E} \left| \int_T f_n dx^* m \right| < \varepsilon$ for $n \geq n_o$, which contradicts (6.4). Hence u is weakly compact. \square

Theorem 13. *Suppose the quasicomplete lcHs X contains no copy of c_o . Then every continuous linear map $u : C_o(T) \rightarrow X$ is weakly compact. If ω is the set \mathbb{N} endowed with the discrete topology and if X is a quasicomplete lcHs such that every continuous linear map $u : C_o(\omega) \rightarrow X$ is weakly compact, then X contains no copy of c_o . Consequently, a quasicomplete lcHs X contains no copy of c_o if and only if each continuous linear map $u : C_o(T) \rightarrow X$ is weakly compact for every locally compact Hausdorff space T .*

Proof. Suppose X contains no copy of c_o and suppose $u : C_o(T) \rightarrow X$ is a continuous linear map. Let m be the representing measure of u . Given an open Baire set U in T , let (K_n) and (g_n) be chosen as in the proof of (i) \Rightarrow (ii) of Theorem 12. Then by the Lebesgue dominated convergence theorem we have

$$x^*m(U) = \lim_n \int_T g_n dx^*m = \sum_{n=1}^{\infty} \int_T (g_{n+1} - g_n) dx^*m + \int_T g_1 dx^*m$$

for each $x^* \in X^*$. Since $g_{n+1} - g_n \geq 0$ for each n , it follows that

$$\begin{aligned} \sum_{n=1}^{\infty} \left| \int_T (g_{n+1} - g_n) dx^*m \right| &\leq \sum_{n=1}^{\infty} \int_T (g_{n+1} - g_n) d|x^*m| \\ &= \lim_n \int_T g_{n+1} d|x^*m| - \int_T g_1 d|x^*m| \leq 2|x^*m|(U) < \infty. \end{aligned}$$

On the other hand, by Theorem 1(iii), $\int_T (g_{n+1} - g_n) dx^*m = x^*u(g_{n+1} - g_n)$, and hence it follows that $\sum_{n=1}^{\infty} |x^*(u(g_{n+1}) - u(g_n))| < \infty$ for each $x^* \in X^*$. Since X contains no copy of c_o , by Theorem 4 of Tumarkin [Tu] we conclude that $\sum_{n=1}^{\infty} (u(g_{n+1}) - u(g_n))$ is convergent in the topology of X and thus $\lim_n u(g_n) = x$ exists in X . Then $x^*m(U) = \lim_n \int_T g_n dx^*m = \lim_n x^*u(g_n) = x^*x$ for each $x^* \in X^*$, and consequently $m(U) = x \in X$. Therefore, u is weakly compact by Theorem 3(vii).

Suppose each continuous linear map $u : C_o(\omega) \rightarrow X$ is weakly compact, and suppose τ is the topology of X . Let (x_n) be a sequence of vectors in X such that $\sum_{n=1}^{\infty} |x^*(x_n)| < \infty$ for each $x^* \in X^*$. Let us define $u(\chi_{\{n\}}) = x_n$ for each n and extend u linearly on the set S of all scalar functions of the form $\sum_{i=1}^k \alpha_i \chi_{\{n_i\}}$, where S is endowed with the supremum norm. By hypothesis on (x_n) , the set $\{uf : f \in S, \|f\|_{\mathbb{N}} \leq 1\}$ is weakly bounded in X and hence is τ -bounded. Then by Theorem 1.32 of Rudin [R], u is continuous on S . Since $S = C_c(\omega)$ is norm dense in $C_o(\omega)$ and X is quasicomplete, u admits a unique continuous linear extension to the whole of $C_o(\omega)$; let us denote the extension too by u . Then by hypothesis u is weakly compact. Let m be the representing measure of u . Since $u^{**}|_{C_o(\omega)} = u$, it follows that $m(F) = u(\chi_F) = \sum_{n \in F} x_n$, for each finite set $F \subset \mathbb{N}$. Since u is weakly compact, by Theorem 2(ii) the range of m is contained in X ; and by Theorem 6(xx), m is Borel inner regular. Thus, given a subset U of \mathbb{N} , $\varepsilon > 0$ and a τ -continuous seminorm p on X , there exists a finite set $F \subset U$ such that

$$p(m(U) - \sum_{n \in G} x_n) < \varepsilon$$

for all finite sets G with $F \subset G \subset U$. Therefore, $m(U) = \sum_{n \in U} x_n$, where the series is unconditionally τ -convergent in X . In particular, $\sum_{n=1}^{\infty} x_n$ is unconditionally τ -convergent; and thus by Theorem 4 of Tumarkin [Tu], X contains no copy of c_o . \square

Remark 10. The first part of the above theorem was proved by Pelczyński [Pe] when T is compact and X is a Banach space, and was later extended by Thomas [Th] for the general case in an equivalent form. Both proofs use the technique of reduction to the compact metrizable case. Thanks to our Theorem 3(vii) we are able to provide a direct proof without such a reduction. The second part of the above theorem is essentially Theorem 5.3 (bounded Radon measure case) of Thomas [Th], where the proof is based on the theory of Radon vector measures.

Added in proof. In a recent paper submitted for publication, the author has given an alternative proof to obtain the said characterizations of weakly compact operators on $C_0(T)$. The new proof is based on the regular Borel extension of σ -additive X -valued Baire measures on T . \square

REFERENCES

- [BDS] R.G. Bartle, N. Dunford, and J.T. Schwartz, *Weak compactness and vector measures*, Canad. J. Math. **7**, (1955), 289–305. MR **16**:1123c
- [DU] J. Diestel, and J.J. Uhl, *Vector Measures*, Survey No.15, Amer. Math. Soc. Providence, R.I., (1977). MR **56**:12216
- [Di] N. Dinculeanu, *Vector Measures*, Pergamon Press, New York, (1967). MR **34**:6011
- [DK] N. Dinculeanu, and I. Kluvánek, *On vector measures*, Proc. London Math. Soc. **17**, (1967), 505–512. MR **35**:5571
- [DL] N. Dinculeanu, and P.W. Lewis, *Regularity of Baire measures*, Proc. Amer. Math. Soc. **26**, (1970), 92–94. MR **41**:5588
- [Do] I. Dobrakov, *On integration in Banach spaces, IV*, Czech. Math. J. **30**, (1980), 259–279. MR **81m**:28006
- [E] R. E. Edwards, *Functional Analysis, Theory and Applications*, Holt, Rinehart and Winston, New York, (1965). MR **36**:4308
- [G] A. Grothendieck, *Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$* , Canad. J. Math. **5**, (1953), 129–173. MR **15**:438b
- [Ha] P. R. Halmos, *Measure Theory*, Van Nostrand, New York, (1950). MR **11**:504d
- [Ho] J. Horváth, *Topological Vector Spaces and Distributions*, Addison-Wesley, Reading, (1966). MR **34**:4863
- [K] I. Kluvánek, *Characterization of Fourier-Stieltjes transforms of vector and operator valued measures*, Czech. Math. J. **17**, (1967), 261–277. MR **37**:6430
- [M] C. W. McArthur, *On a theorem of Orlicz and Pettis*, Pacific J. Math., **22**, (1967), 297–302. MR **33**:4702
- [Pa1] T.V. Panchapagesan, *On complex Radon measures I*, Czech. Math. J. **42**, (1992), 599–612. MR **94e**:28014
- [Pa2] ———, *On complex Radon measures II*, Czech. Math. J. **43**, (1993), 65–82. MR **94i**:28011
- [Pa3] ———, *Abstract regularity of additive and σ -additive group-valued set functions*, Math. Slovaca, **45**, (1995), 381–393. MR **97c**:28027
- [Pa4] ———, *Baire and σ -Borel characterizations of weakly compact sets in $C_o(T)$* , Trans. Amer. Math. Soc. **350** (1998), 4839–4847.
- [Pa5] ———, *On Radon vector measures I*, submitted.
- [Pe] A. Pelczyński, *Projections in certain Banach spaces*, Studia Math. **19**, (1960), 209–228. MR **23**:A3441
- [R] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, (1973). MR **51**:1315
- [Th] E. Thomas, *L'intégration par rapport a une mesure de Radon vectorielle*, Ann. Inst. Fourier (Grenoble) **20**, (1970), 55–191. MR **57**:3348
- [Tu] Ju. B. Tumarkin, *On locally convex spaces with basis*, Soviet Math. Dokl. **11**, (1970), 1672–1675. MR **42**:6575

DEPARTAMENTO DE MATEMÁTICAS, FACULTAD DE CIENCIAS, UNIVERSIDAD DE LOS ANDES,
MÉRIDA, VENEZUELA

E-mail address: panchapa@ciens.ula.ve